ON THE KÜNNETH THEOREM

BY

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Introduction. The Künneth theorem asserts that the homology $H(X\otimes Y)$ of the product of two free abelian positively graded chain complexes is isomorphic to $\mathrm{Tor}(HX, HY)=HX\otimes HY+\mathrm{Tor}_1(HX, HY)$. This isomorphism however gives no insight into the effect of chain maps; that is to say, $H(f\otimes g)$ cannot be computed in terms of Hf and Hg. Alternatively the difficulty may be expressed by saying that the isomorphism in question is not canonical.

It appears that in order to produce a functorial Künneth theorem some additional invariant of a chain complex is needed, and indeed Bockstein [1] and Palermo [4] have produced such theorems using as the starting point the homology or cohomology spectra.

In this paper we adopt the alternate course of introducing as an invariant the affine space of orientations (§2) of a chain complex. Having defined the notion of skew-isomorph (§3) of a group with respect to such an affine space we then see that the homology of the product, as well as cohomology groups, groups of homotopy classes and so forth, are just skew-isomorphs of those produced by the usual functors of homological algebra.

In §6 we use the same device to sharpen the notion of derived functor, and with these sharpened functors we are able to compute the space of orientations of a product complex (§7). There is an obvious application, which however we omit, to the triple (etc.) product studied by MacLane [3].

The external product in cohomology, and also the cup-product, have straightforward expressions in the language here adduced. These are given in §8.

1. **Unfolding chain complexes.** By *chain complexes* we shall mean free abelian graded groups with derivations of degree -1. We shall further require that these groups be graded by nonnegative degrees, i.e., that their homogeneous components of negative degree be 0. Some of our results remain valid without the latter restriction; the reader may easily supply the generalizations for himself.

We shall also have occasion to consider *bicomplexes*, by which we understand free abelian bigraded groups $A=\sum_{p\geq 0, q\geq 0} A_{pq}$ with derivations of degree $(0, -1)$. We shall call p the *principal*, q the *resolvent* degree of A_{pq}.

The notions of chain map and chain homotopy of chain complexes are the usual ones; we extend them by analogy to bicomplexes. Observe that in the latter case a chain homotopy of degree (p, q) connects maps of degree $(p, q-1)$.

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If X is a chain complex its *unfolded complex* UX is the bicomplex.

$$(UX)_{p0} = Z_pX,\ (UX)_{p1} = B_pX,\ (UX)_{pq} = 0, \qquad q \neq 0, 1,$$

supplied with the inclusion $\beta X: BX \rightarrow ZX$ as derivation. Notice that HUX is bigraded, but that $H_{*0}UX = HX$, $H_{*q}UX = 0$, $q \neq 0$.

We may indeed make U into a functor by setting

$$(Uf)_{*0} = Zf,\ (Uf)_{*1} = Bf,\ (Uf)_{*q} = 0, \qquad q \neq 0, 1.$$

If f is homogeneous of degree r then Uf is homogeneous of degree $(r, 0)$.

The point of this construction is of course that UX is a projective resolution of HX, and Uf a projective resolution of Hf, for a chain map f. If we write Hom for the group of homotopy classes of chain maps then $\mathrm{Hom}(UX, UX')$ may be identified with the bigraded group $\mathrm{Ext}(HX, HX')$, while the composition $\mathrm{Hom}(UX', UX'') \otimes \mathrm{Hom}(UX, UX') \rightarrow \mathrm{Hom}(UX, UX'')$ is just the composition in the category Ext $\mathcal{G}^\infty$, the extended category of the category of graded abelian groups [2].

We denote by Σ the functor which makes a bigraded group into a graded one by employing the total degree only. Clearly Σ takes bicomplexes into complexes and has the following properties:

$$H\Sigma = \Sigma H, \qquad H\Sigma U = H, \qquad U\Sigma U = U. \tag{1.1}$$

It is also clear that if Y, Y' are bicomplexes then

$$\Sigma\, \mathrm{Hom}(Y, Y') = \mathrm{Hom}(\Sigma Y, \Sigma Y'). \tag{1.2}$$

If A is a bigraded group then A and ΣA have the same elements; we shall on occasion allow ourselves the liberty of writing A instead of ΣA.

2. **Orientations and homotopy-classes of maps.** If X is a chain-complex then $H\Sigma UX = HX$. Thus there are chain maps $f: X \rightarrow \Sigma UX$ such that $Hf = 1: HX$; such maps are of course homotopy equivalences. We denote by ΓX the set of homotopy classes of these maps, and call such homotopy classes *orientations* of X.

We shall see that ΓX is provided with the structure of an affine space over a certain group. To be explicit, let us first define, for any group G, a *G-preaffine space* as a set Γ together with a map $\delta: \Gamma \times \Gamma \rightarrow G$ such that for any $x, y, z \in \Gamma$, $\delta(x, y)\delta(y, z) = \delta(x, z)$. A G-preaffine space (Γ, δ) is *affine* if, first, for any $x, y \in \Gamma$, $\delta(x, y) = 1$ implies $y = x$, and second, for any $x \in \Gamma$, $\{\delta(x, y) \mid y \in \Gamma\} = G$. The notion of G-affine space, then, is equivalent to that of principal homogeneous space of G, the operation being prescribed by the equation $\delta(y, x)x = y$ for $x, y \in \Gamma$.

Now if A is a graded abelian group we define the group xtr A of *extransvections* of A to be $\{1: A + \phi \mid \phi \in \mathrm{Ext}_1^1(A, A)\} \subset \mathrm{Ext}(A, A)$. This is a multiplicative subgroup of the ring $\mathrm{Ext}(A, A)$ which is of course canonically isomorphic to $\mathrm{Ext}_1^1(A, A)$.

PROPOSITION 2.1. *If X is a chain complex then ΓX is an affine space of* xtr HX.

We need only define, for $\xi, \xi' \in \Gamma X$

$$\delta(\xi', \xi) = \xi'\xi^{-1} \in \mathrm{Hom}(\Sigma UX, \Sigma UX) = \Sigma\, \mathrm{Hom}(UX, UX) = \Sigma\, \mathrm{Ext}\,(HX, HX).$$

We shall call the pair $(HX, \Gamma X)$ the *sharp homology* $\mathbf{H}X$ of the chain complex X. If $\phi: X \to X'$ is a homotopy class of maps we shall write, for $\xi \in \Gamma X$, $\xi' \in \Gamma X'$,

$$(2.2) \qquad (\mathbf{H}\phi)(\xi', \xi) = \xi'\phi\xi^{-1} \in \mathrm{Hom}(\Sigma UX, \Sigma UX') = \mathrm{Ext}(HX, HX');$$

if f is a map in the homotopy class ϕ we shall also write $\mathbf{H}f(\xi', \xi) = \mathbf{H}\phi(\xi', \xi)$.

THEOREM 2.3. *If $\phi: X \to X'$ then $\mathbf{H}\phi: \Gamma X' \times \Gamma X \to \mathrm{Ext}(HX, HX')$ satisfies the condition*

$$(2.3.1) \qquad \mathbf{H}\phi(\xi_1', \xi_1) = \delta(\xi_1', \xi')\mathbf{H}\phi(\xi', \xi)\delta(\xi, \xi_1)$$

for $\xi, \xi_1 \in \Gamma X$, $\xi', \xi_1' \in \Gamma X'$. If also $\phi': X' \to X''$ and $\xi'' \in \Gamma X''$ then

$$(2.3.2) \qquad \mathbf{H}(\phi'\phi)(\xi'', \xi) = \mathbf{H}\phi'(\xi'', \xi')\mathbf{H}\phi(\xi', \xi).$$

If $\theta: \Gamma X' \times \Gamma X \to \mathrm{Ext}(HX, HX')$ satisfies 2.3.1 then there is exactly one homotopy class $\phi: X \to X'$ such that $\mathbf{H}\phi = \theta$.

We shall call $\mathbf{H}\phi: \Gamma X' \times \Gamma X \to \mathrm{Ext}(HX, HX')$ the *sharp homology* of the homotopy class ϕ.

3. **Skew-isomorphs.** Suppose that A is an abelian group, G a group which operates on A and Γ a G-preaffine space. We define the *skew-isomorph* $\Gamma \diamond A$ of A to be the group consisting of maps $\theta: \Gamma \to A$ such that

$$(3.1) \qquad \theta\gamma' = \delta(\gamma', \gamma)\theta\gamma, \qquad \gamma, \gamma' \in \Gamma,$$

with group operation given by $(\theta + \theta')\gamma = \theta\gamma + \theta'\gamma$.

For any $\gamma \in \Gamma$, $a \in A$ there is a unique $\theta \in \Gamma \diamond A$, denoted by $\langle \gamma; a \rangle$, such that $\theta\gamma = a$. Thus $\Gamma \diamond A$ is isomorphic to A. It is *not* however canonically isomorphic to A.

Observe that the following relationship is an immediate consequence of 3.1:

$$\langle \gamma; a \rangle = \langle \gamma_1; \delta(\gamma_1, \gamma)a \rangle, \qquad \gamma, \gamma_1 \in \Gamma, a \in A.$$

If A is graded and the operations of G preserve the degree then $\Gamma \diamond A$ inherits the gradation.

For example, suppose that $0 \to A' \to^{a'} A \to^{a''} A'' \to 0$ is a short exact sequence of abelian groups which splits, and denote by Ψ the set of left-splittings, i.e., of homomorphisms $\alpha: A \to A'$ such that $\alpha a' = 1: A'$. If $\alpha \in \Psi$ we define the conjugate right-splitting $\alpha^*: A'' \to A$ by the conditions $a''\alpha^* = 1: A$, $\alpha\alpha^* = 0$.

The group $\operatorname{tr}(A', A'')$ of *transvections* of $A' \oplus A''$ consists of the automorphisms $(x', x'') \rightarrow (x' + \phi x'', x'')$ where $\phi: A'' \rightarrow A'$ and is of course canonically isomorphic to $\operatorname{Hom}(A'', A')$. We make Ψ a $\operatorname{Hom}(A'', A')$-affine space, and hence a $\operatorname{tr}(A', A'')$-affine space, by setting $\delta(\alpha, \alpha') = \alpha'\alpha^*$. We have then the following result:

(3.2) A is canonically isomorphic to $\Psi \diamond (A' \oplus A'')$. The isomorphism is given by $x \rightarrow \langle \alpha; (\alpha x, a'' x) \rangle$ for $x \in A$, $\alpha \in \Psi$.

For a second example, suppose X and X' are chain complexes. Then $(\operatorname{xtr} HX') \times (\operatorname{xtr} HX)$ operates on $\operatorname{Ext}(HX, HX')$ by means of the composition in $\operatorname{Ext} \mathcal{G}^\infty$. Theorem 2.2 clearly implies the following result.

PROPOSITION 3.3. *If X, X' are chain complexes then* $\operatorname{Hom}(X, X')$ *is canonically isomorphic to* $(\Gamma X' \times \Gamma X) \diamond \operatorname{Ext}(HX, HX')$.

A third example is the following. Suppose A, A' are abelian groups graded by nonnegative degrees. If $\alpha \in \operatorname{xtr} A$, $\alpha' \in \operatorname{xtr} A'$ then $\operatorname{Tor}(\alpha, \alpha'): \operatorname{Tor}(A, A') \rightarrow \operatorname{Tor}(A, A')$ is an automorphism of total degree 0, so that $\operatorname{xtr} A \times \operatorname{xtr} A'$ operates on $\operatorname{Tor}(A, A')$ preserving the total degree. If Γ, Γ' are respectively $\operatorname{xtr} A$, $\operatorname{xtr} A'$-affine then $\Gamma \times \Gamma'$ is $\operatorname{xtr} A \times \operatorname{xtr} A'$-affine and there is defined a group $(\Gamma \times \Gamma') \diamond \operatorname{Tor}(A, A')$ which is noncanonically isomorphic to $\Sigma \operatorname{Tor}(A, A')$, i.e., to $\operatorname{Tor}(A, A')$ graded by the total degree.

4. **The functorial Künneth theorem.** If Y and Y' are bicomplexes we define $Y \otimes Y'$ by $(Y \otimes Y')_{mn} = \sum_{i+j=m, p+q=n} Y_{ip} \otimes Y'_{jq}$ with derivation $\partial(y \otimes y') = \partial y \otimes y' + (-1)^{i+p} y \otimes \partial y'$ for $y \in Y_{ip}$. Notice that this product commutes with Σ, i.e., $\Sigma(Y \otimes Y') = \Sigma Y \otimes \Sigma Y'$.

If X and X' are chain complexes then UX, UX' are projective resolutions of HX, HX' and thus $H(UX \otimes UX') = \operatorname{Tor}(HX, (HX')$, so that $H(\Sigma UX \otimes \Sigma UX') = \Sigma \operatorname{Tor}(HX, HX')$.

THEOREM 4.1. *If X, X' are chain complexes then $H(X \otimes X')$ is canonically isomorphic to*

$$(\Gamma X \times \Gamma X') \diamond \operatorname{Tor}(HX, HX').$$

If $f: X \rightarrow X_1$, $f': X' \rightarrow X_1'$ then, making the identification by means of this canonical isomorphism,

$$(4.1.1) \quad H(f \otimes f')\langle \xi', \xi'; a \rangle = \langle \xi_1, \xi_1'; \operatorname{Ext}(\mathbf{H}f(\xi_1, \xi), \mathbf{H}f(\xi_1', \xi'))a \rangle$$

for $\xi \in \Gamma X$, $\xi' \in \Gamma X'$, $\xi_1 \in \Gamma X_1$, $\xi_1' \in \Gamma X_1'$, $a \in \operatorname{Tor}(HX, HX')$.

The isomorphism $\Phi = \Phi(X, X'): H(X \otimes X') \rightarrow (\Gamma X \times \Gamma X') \diamond \operatorname{Tor}(HX, HX')$ is given by

$$(\Phi x)(\xi, \xi') = H(\xi \otimes \xi')x, \qquad \xi \in \Gamma X, \ \xi' \in \Gamma X', \ x \in H(X \otimes X').$$

Observe that if $\xi_1 \in \Gamma X$, $\xi_1' \in \Gamma X'$ then

$$H(\xi_1 \otimes \xi_1')x = H(\xi_1\xi^{-1} \otimes \xi_1'\xi'^{-1})[\Phi x(\xi, \xi')]$$
$$= \mathrm{Ext}(\delta(\xi_1, \xi), \delta(\xi_1', \xi'))[\Phi x(\xi, \xi')].$$

Formula 4.1.1 is proved analogously.

Similar techniques may be applied to the universal coefficient theorem. We have the following results.

THEOREM 4.2. *If X is a chain complex and A an abelian group then $H(X\otimes A)$ is canonically isomorphic to $\Gamma X \diamond \mathrm{Tor}(HX, A)$. The cohomology $H^*(X; A)$ is canonically isomorphic to $\Gamma X \diamond \mathrm{Ext}(HX, A)$. The effect of chain maps is, in each case, given by the analogue of* 4.1.1. *For example if $f: X\to X'$, $\xi\in\Gamma X$, $\xi'\in\Gamma X'$, $\alpha\in\mathrm{Ext}(HX', A)$ then*

$$H^*f\langle\xi'; \alpha\rangle = \langle\xi; \alpha\mathbf{H}f(\xi', \xi)\rangle.$$

The effect of coefficient homomorphisms and the Bockstein homomorphism may easily be evaluated in terms of the above representations of homology and cohomology. We have (confining ourselves to cohomology) the following result.

THEOREM 4.3. *If X is a chain complex, A and A' are abelian groups and $a: A\to A'$ is a homomorphism then, for $\xi\in\Gamma X$, $\alpha\in\mathrm{Ext}(HX, A)$,*

$$(H_*a)\langle\xi; \alpha\rangle = \langle\xi; a\alpha\rangle.$$

If $\mathbf{A}=(0\to A'\to A\to A''\to 0)$ is an exact sequence of abelian groups with extension class $\Delta\mathbf{A}\in\mathrm{Ext}^1(A'', A')$ then the Bockstein operation $bA: H^*(X; A'') \to H^*(X; A')$ is given by

$$bA\langle\xi; \alpha''\rangle = \langle\xi; (\Delta\mathbf{A})\alpha''\rangle, \qquad \xi\in\Gamma X, \alpha''\in\mathrm{Ext}(HX, A'').$$

Theorems 4.2 and 4.3 constitute a computation of the Bockstein cohomology spectrum in terms of the sharp homology. It has been observed (e.g. [4]) that functorial Künneth theorems may also be stated in terms of the homology or cohomology spectra.

5. **An alternate description of orientations.** Inasmuch as the affine space ΓX of orientations of a chain complex X is proposed as a fundamental invariant of X it may be appropriate to give an alternate description of ΓX in what may be somewhat more familiar form.

We have seen (4.2) that $H^*(X; A)$ may be identified with $\Gamma X \diamond \mathrm{Ext}(HX, A)$. Now the subgroup $\mathrm{Ext}^1(HX, A)$ of $\mathrm{Ext}(HX, A)$ is invariant under the operation of xtr HX. Thus the exact sequence $0\to\mathrm{Ext}^1(HX, A) \to\mathrm{Ext}(HX, A)\to\mathrm{Hom}(HX, A)\to 0$ is also invariant and indeed xtr HX operates by transvections on $\mathrm{Ext}(HX, A)=\mathrm{Ext}^1(HX, A)\oplus\mathrm{Hom}(HX, A)$. Thus the sequence

$$0\to \Gamma X\diamond \mathrm{Ext}_1^1(HX, A)\to \Gamma X\diamond(\Sigma\,\mathrm{Ext}(HX, A))_0\to\Gamma X\diamond\mathrm{Hom}_0(HX, A)\to 0$$

is exact and may be identified canonically with the usual universal coefficient sequence

$$0 \to \mathrm{Ext}_1^1(HX, A) \xrightarrow{\mathfrak{d}^*} H^0(X; A) \xrightarrow{\mathfrak{h}^*} \mathrm{Hom}_0(HX, A) \to 0.$$

A similar construction gives the Künneth exact sequence.

Now the map $\xi \to r\xi = \langle \xi; 1: HX\rangle \in H^0(X; HX)$ maps ΓX onto the subset of $H^0(X; HX)$ consisting of those elements α such that $\mathfrak{h}^*\alpha = 1: HX$, and indeed we have, for $\xi, \xi' \in \Gamma X$ and $\delta(\xi, \xi') = 1 + \phi$, $\phi \in \mathrm{Ext}_1^1(HX, HX)$,

$$\begin{aligned} r\xi - r\xi' &= \langle \xi; 1\rangle - \langle \xi'; 1\rangle = \langle \xi; 1\rangle - \langle \xi; \delta(\xi, \xi')\rangle \\ &= \mathfrak{d}^*\phi. \end{aligned} \tag{5.1}$$

If we identify ξ with $r\xi$ then we have the following result.

PROPOSITION 5.2. *If X is a chain complex then ΓX is the subset of $H^0(X; HX)$ consisting of elements ξ such that $\mathfrak{h}^*\xi = 1: HX$. The affine-space structure is given by*

$$\delta(\xi, \xi') = 1: HX + \mathfrak{d}^{*-1}(\xi - \xi') \in \mathrm{xtr}\, HX.$$

For the most part however it is more convenient to use the characterization given in §2.

6. **The functor Tor; sharp derived functors.** If the homology of a chain complex X vanishes except in one degree then xtr HX is trivial and ΓX is more or less irrelevant as an invariant. This is the case with a projective resolution, but is *not* the case with the product of two projective resolutions. If A and A' are abelian groups then $H(\mathbf{PR}A \otimes \mathbf{PR}A') = \mathrm{Tor}(A, A')$ has in general two nontrivial degrees and thus $\Gamma(\mathbf{PR}A \otimes \mathbf{PR}A') = \Theta(A, A')$ is a nontrivial invariant of the pair A, A'. We denote $(\mathrm{Tor}(A, A'), \Theta(A, A'))$ $= \mathbf{H}(\mathbf{PR}A \otimes \mathbf{PR}A')$ by $\mathbf{Tor}(A, A')$.

If $\phi \in \mathrm{Ext}(A, A_1)$, $\phi' \in \mathrm{Ext}(A', A_1')$, i.e., $\phi: \mathbf{PR}A \to \mathbf{PR}A_1$ and $\phi': \mathbf{PR}A' \to \mathbf{PR}A_1'$ are homotopy classes of maps, then $H(\phi \otimes \phi') = \mathrm{Tor}(\phi, \phi'): \mathrm{Tor}(A, A') \to \mathrm{Tor}(A_1, A_1')$. But we have further for $t \in \Theta(A, A')$, $t_1 \in \Theta(A_1, A_1')$

$$\mathbf{H}(\phi \otimes \phi')(t_1, t) = t_1(\phi \otimes \phi')t^{-1} \in \mathrm{Ext}(\mathrm{Tor}(A, A'), \mathrm{Tor}(A_1, A_1')) \tag{6.1}$$

of which $\mathrm{Tor}(\phi, \phi')$ is a projection. We write

$$\mathbf{H}(\phi \otimes \phi') = \mathbf{Tor}(\phi, \phi'): \Theta(A_1, A_1') \times \Theta(A, A') \to \mathrm{Ext}(\mathrm{Tor}(A, A'), \mathrm{Tor}(A_1, A_1')).$$

This should be thought of as a "sharpening" of the usual torsion functor.

We shall call **Tor** the *sharp torsion functor*. Similar constructions, applied when appropriate to other functors, yield the *sharp derived functors*.

It will be convenient to define also a product intermediate between Tor and **Tor.** If $\pi: \mathrm{Ext}(\mathrm{Tor}(A, A'), \mathrm{Tor}(A_1, A_1')) \to \mathrm{Ext}(\mathrm{Tor}(A, A'), A_1 \otimes A_1')$ is the projection then for $\phi \in \mathrm{Ext}(A, A_1)$, $\phi' \in \mathrm{Ext}(A_1, A_1')$, the composition

$$\pi\,\mathbf{Tor}(\phi, \phi')(t_1, t)\colon \Theta(A_1, A_1') \times \Theta(A, A') \to \operatorname{Ext}\,(\operatorname{Tor}(A, A'), A_1 \otimes A_1')$$

is independent of $t_1 \in \Theta(A_1, A_1')$. We shall write

$$\mathbf{Tor}^b(\phi, \phi')_t = \mathbf{Tor}(\phi, \phi')(t_1, t), \qquad t \in \Theta(A, A'),\ t_1 \in \Theta(A_1, A_1').$$

These construction must also be applied in the case that A, A' are graded abelian groups. Here $\mathbf{PR}A$, $\mathbf{PR}A'$ are bicomplexes and $U(\mathbf{PR}A \otimes \mathbf{PR}A')$ is triply graded with derivation of degree $(0, 0, -1)$. An orientation of $\mathbf{PR}A \otimes \mathbf{PR}A'$ is a homotopy class t: $\mathbf{PR}A \otimes \mathbf{PR}A' \to \Sigma U(\mathbf{PR}A \otimes \mathbf{PR}A')$ and the set $\Theta(A, A')$ of orientations is xtr $\operatorname{Tor}(A, A')$-affine where xtr $\operatorname{Tor}(A, A') = \{1 + \phi \mid \phi \in \operatorname{Ext}^1_{01}(\operatorname{Tor}(A, A'), \operatorname{Tor}(A, A'))\}$. Modulo these observations the treatment is the same as in the ungraded case.

7. **Orientations of a product complex.** We shall amplify the Künneth Theorem 4.1 above by investigating, for chain complexes X, X', the affine space $\Gamma(X \otimes X')$. In order to do this we define for $\xi \in \Gamma X$, $\xi' \in \Gamma X'$, $t \in \Theta(HX, HX')$ an orientation $\mathrm{t}(\xi, \xi', t)$ of $X \otimes X'$ by requiring commutativity in the diagram

$$\begin{tikzcd}
X \otimes X' \arrow[r, "\xi \otimes \xi'"] \arrow[d, "\mathrm{t}(\xi, \xi', t)"] & \Sigma UX \otimes \Sigma UX' \arrow[d, "t"] \\
\Sigma U(X \otimes X') \arrow[r, "\Sigma U(\xi \otimes \xi')"] & \Sigma U(\Sigma XU \otimes \Sigma UX').
\end{tikzcd}$$

Since $H(\xi \otimes \xi') = H\Sigma U(\xi \otimes \xi')$ and $Ht = 1\colon \operatorname{Tor}(HX, HX')$ it follows that $H\mathrm{t}(\xi, \xi', t) = 1\colon H(X \otimes X')$, which is of course the defining condition for an orientation.

LEMMA 7.1. *If also* $\xi_1 \in \Gamma X_1$, $\xi_1' \in \Gamma X_1'$, $t_1 \in \Theta(HX_1, HX_1')$, $f\colon X \to X_1$ *and* $f'\colon X' \to X_1'$ *then*

$$\begin{aligned}
&\mathbf{H}(f \otimes f')(\mathrm{t}(\xi_1, \xi_1', t_1), \mathrm{t}(\xi, \xi', t)) \\
&\qquad = H(\xi \otimes \xi')^{-1}[\mathbf{Tor}(\mathbf{H}f(\xi_1, \xi), \mathbf{H}f'(\xi_1', \xi'))(t_1, t)]H(\xi \otimes \xi').
\end{aligned}$$

For

$$\begin{aligned}
&\mathbf{H}(f \otimes f')(\mathrm{t}(\xi_1, \xi_1', t_1), \mathrm{t}(\xi, \xi', t)) \\
&\qquad = \mathrm{t}(\xi_1, \xi_1', t_1)(f \otimes f')\mathrm{t}(\xi, \xi', t)^{-1} \\
&\qquad = \Sigma U(\xi_1 \otimes \xi_1')^{-1} t_1 (\xi_1 \otimes \xi_1')(f \otimes f')(\xi \otimes \xi')^{-1} t^{-1} \Sigma U(\xi \otimes \xi') \\
&\qquad = H(\xi_1 \otimes \xi_1')^{-1}[\mathbf{Tor}(\mathbf{H}f(\xi_1, \xi), \mathbf{H}f'(\xi_1', \xi'))(t_1, t)]H(\xi \otimes \xi')
\end{aligned}$$

by 2.2 and 6.1, recalling that $\Sigma U(\xi \otimes \xi')$ is just a projective resolution of $H(\xi \otimes \xi')$.

As a special case we have the following result.

COROLLARY 7.2. *If* $\xi, \xi_1 \in \Gamma X$, $\xi', \xi_1' \in \Gamma X'$, $t, t_1 \in \Theta(HX, HX')$ *then*

$$\delta(\mathrm{t}(\xi_1, \xi_1', t_1), \mathrm{t}(\xi, \xi', t)) = H(\xi_1 \otimes \xi_1')^{-1}[\mathbf{Tor}(\delta(\xi_1, \xi), \delta(\xi_1', \xi'))(t_1, t)]H(\xi \otimes \xi').$$

From these data $\Gamma(X \otimes X')$ may easily be computed as an identification space of $\Gamma X \times \Gamma X' \times \Theta(HX, HX') \times$ xtr $\mathrm{Tor}(HX, HX')$. We shall omit this as being of secondary interest; the important thing is to have constructed the orientations $\mathrm{t}(\xi, \xi', t)$.

8. **Cohomology of a product; external and cup-products.** If X and X' are chain complexes and A is an abelian group the cohomology $H^*(X \otimes X'; A)$ may be computed by 4.2 from $H(X \otimes X') = (\Gamma X \times \Gamma X') \diamond \mathrm{Tor}(HX, HX')$ and $\Gamma(X \otimes X')$. We shall give here an alternate, somewhat less laborious, computation.

Consider the maps

$$H^*(X \otimes X'; A) \xleftarrow{H^*(\xi \otimes \xi')} H^*(\Sigma UX \otimes \Sigma UX'; A) \xleftarrow{H^* t} H^*(\Sigma U(\Sigma UX \otimes \Sigma UX'); A),$$

where $\xi \in \Gamma X$, $\xi' \in \Gamma X'$, $t \in \Theta(HX, HX')$. The group on the right is just $\mathrm{Ext}(\mathrm{Tor}(HX, HX'), A)$ and both maps are isomorphisms. Extending our notation by analogy we write, for α in this group, $\langle \xi, \xi', t; \alpha\rangle = H^*(\xi \otimes \xi') \cdot (H^* t)\alpha$.

If $f: X_1 \to X$, $f': X_1' \to X'$, $\xi_1 \in \Gamma X_1 \xi_1' \in \Gamma X_1'$, $t' \in \Theta(HX_1, HX_1')$ it is easy to see that

$$(8.1) \qquad H^* f\langle \xi, \xi', t; \alpha\rangle = \langle \xi_1, \xi_1', t_1; \alpha \, \mathbf{Tor}(\mathbf{H}f(\xi, \xi_1), \mathbf{H}f(\xi', \xi_1'))(t_1, t)\rangle$$

and, taking $X = X_1$, $X' = X_1'$ and f, f' the identity maps,

$$(8.2) \qquad \langle \xi, \xi', t; \alpha\rangle = \langle \xi_1, \xi_1', t_1; \alpha \, \mathbf{Tor}(\delta(\xi, \xi_1), \delta(\xi', \xi_1'))(t_1, t)\rangle.$$

We have thus proved the following result.

THEOREM 8.3. *If X, X' are chain complexes and A is an abelian group then $H^*(X \otimes X'; A)$ is canonically isomorphic to*

$$(\Gamma X \times \Gamma X' \times \Theta(HX, HX')) \diamond \mathrm{Ext}(\mathrm{Tor}(HX, HX'), A)$$

where $\Gamma X \times \Gamma X' \times \Theta(HX, HX')$ is an $\mathrm{Ext}(\mathrm{Tor}(HX, HX'), \mathrm{Tor}(HX, HX'))$-preaffine space under the operation

$$\delta((\xi_1, \xi_1', t_1), (\xi, \xi', t)) = \mathbf{Tor}(\delta(\xi_1, \xi), \delta(\xi_1', \xi'))(t_1, t).$$

Using this expression for $H^*(X \otimes X'; A)$ we may evaluate the "external" product $\otimes: H^*(X; A) \otimes H^*(X'; A') \to H^*(X \otimes X'; A \otimes A')$ as follows.

THEOREM 8.4. *If $\xi \in \Gamma X$, $\alpha \in \mathrm{Ext}(HX; A)$, $\xi' \in \Gamma X'$, $\alpha' \in \mathrm{Ext}(HX', A')$ and $t \in \Theta(HX, HX')$ then*

$$\langle \xi; \alpha\rangle \otimes \langle \xi'; \alpha'\rangle = \langle \xi, \xi', t; \mathbf{Tor}^b(\alpha, \alpha')_t\rangle.$$

This results from commutativity in the following diagram:

$$\begin{array}{ccc}
\mathrm{Ext}\,(HX,\,A)\otimes\mathrm{Ext}(HX',\,A') & \longrightarrow & \mathrm{Ext}(\mathrm{Tor}(HX,HX'),\,A\otimes A') \\
\downarrow = & & \downarrow H^*t \\
H^*(\Sigma UX;\,A)\otimes H^*(\Sigma UX';\,A') & \xrightarrow{\otimes} & H^*(\Sigma UX\otimes\Sigma\, UX';\,A\otimes A') \\
\downarrow H^*\xi\otimes H^*\xi' & & \downarrow H^*(\xi\otimes\xi') \\
H^*(X:A)\otimes H^*(X':A') & \xrightarrow{\otimes} & H^*(X\otimes X':A\otimes A')
\end{array}$$

where the map in the top row is $\alpha\otimes\alpha'\to\mathbf{Tor}^b(\alpha,\,\alpha')_t$.

We have here tacitly made use of the canonical isomorphism of $H^*(X;B)$ with $\mathrm{Hom}(X,\,\mathbf{PR}B)$. Interpreting cohomology classes in the latter way we arrive at a sharpening of the external product:

(8.5) $\quad H^*(X:A)\otimes H^*(X',\,A')\to\Theta(A,\,A')\diamond H^*(X\otimes X':\mathrm{Tor}(A,\,A'))$

of which the usual product is just a projection. This is of course given by

$$\langle\xi;\,\alpha\rangle\otimes'\langle\xi';\,\alpha'\rangle = \langle t_1;\,\langle\xi,\,\xi',\,t;\,\mathbf{Tor}(\alpha,\,\alpha')(t_1,\,t)\rangle\rangle$$

for $t_1\in\Theta(A,\,A')$.

We conclude with a formula for the cup-product, which is defined for a complex X provided with a diagonal map $\omega\colon X\to X\otimes X$ and is just $(H^*\omega)\otimes=\cup\colon H^*(X;\,A)\otimes H^*(X;\,A')\to H^*(X;\,A\otimes A')$.

THEOREM 8.6. *If X is provided with a diagonal map ω, and if $\xi\in\Gamma X$, $\alpha\in\mathrm{Ext}(HX,\,A)$, $\alpha'\in\mathrm{Ext}(HX,\,A')$, $t\in\Theta(HX,\,HX)$ then*

$$\langle\xi,\,\alpha\rangle\cup\langle\xi,\,\alpha'\rangle = \langle\xi;\,\mathbf{Tor}^b\,\alpha,\,\alpha')_t H_{\xi,t}\,\omega\rangle$$

where

$$\mathbf{H}_{\xi,t}\,\omega = H(\xi\otimes\xi)\mathbf{H}\omega(\mathrm{t}(\xi,\,\xi,\,t),\,\xi)\in\mathrm{Ext}(HX,\,\mathrm{Tor}(HX,\,HX))$$

is an invariant of ω alone.

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